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Resistance of the Edwards walk

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Abstract. An analytical and numerical study of the scaling of Edwards random walks, defined as the zero-component limit ($n \to 0$) of an n-component $g_0|\boldsymbol{\phi}^2|^2$ theory, is done in three dimensions. The Hausdorf dimension ($1/\nu$) and the resistance exponent (x) is calculated using renormalisation group methods in ($4-\varepsilon$) dimensions. The numerical study is done by defining a Monte Carlo algorithm on the cubic lattice to generate ensembles of Edwards walks.

1. Introduction

The relationship between $g_0|\boldsymbol{\phi}^2|^2$ theories ($\boldsymbol{\phi} \in \mathbb{R}^n$) in the zero-component limit ($n \to 0$) and theories of random walks has received considerable attention in recent years [1, 2]. The use of random walks as a laboratory to study (especially the renormalisation of) $g_0|\boldsymbol{\phi}^2|^2$ theory and, in the more general context, phase transitions is consequently firmly established in the literature ([1, 2] and references therein). The motivation for this paper is found in the general study of the scaling properties of random objects, and we shall limit ourselves here to a case study of the Edwards walk model and its resistance exponent, using in this case both a theoretical and numerical approach.

The statistical fractal nature of random walks makes a study of the scaling properties of these objects very natural [3] and the methods used in this paper will centre about this fact. Perhaps the most natural critical exponent associated with fractal objects is the fractal dimension ($1/\nu$) and we shall consider it here also. Define ξ to be the square root of the mean square end-to-end length of an ensemble of random walks. Then

$$\xi \propto N^{\nu} \tag{1.1}$$

where N is the number of links in a walk (on a lattice) or the length of the walk. For Brownian random walks in any number of dimensions it is known that $\nu = \frac{1}{2}$, and for self-avoiding random walks $\nu \approx 0.598$ in three dimensions. In this paper we shall consider this exponent for the Edwards walk in three dimensions.

The electric resistance of random clusters is a well considered topic in the literature, especially in percolation theory [4, 5], but also for Brownian random walk clusters [6]. This paper is a generalisation of the methods in [6] to include the Edwards model, basing the method of investigation on the methods in [5-8]. The resistance (or conductance) exponent (x) for random walks is defined by

$$R \propto N^{x} \tag{1.2}$$

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where R is the average end-to-end resistance of a walk and where $x = 1$ in more than four dimensions, and where logarithmic corrections to (1.2) are expected in four dimensions [6] (it has been conjectured that the Edwards walk and the self-avoiding random walk have trivial limits in four dimensions [1, 2]). The renormalisation group method can be applied to these models in $(4-\varepsilon)$ dimensions to facilitate an ε expansion for the critical exponents [6]. These methods will be applied to calculate the exponent x to first order in ε; higher-order corrections will be published elsewhere.

This paper is organised as follows. In § 2 we briefly discuss the calculation of ν and introduce the calculation of x by the scheme of Wilson and Kogut [8]. A similar calculation was perfomed for Brownian walks in [6] and we relate the problem to that calculation in § 3 when we derive a value for x. In § 4 we define a Monte Carlo (MC) algorithm to generate ensembles of Edwards walks. As a test, the Brownian walk is studied and ν and x are compared with known and published values. These checks should make us confident in the coding of the program. In § 5 we calculate ν and x for Edwards walks and discuss the results. The error analysis is briefly considered and the implications of the results are studied. Our conclusions are presented in § 6 and our results are compared with the findings of related studies [6, 9]. A few suggestions for further study are also made.

2. Preliminary calculations

In this section we first calculate ν for the Edwards model, then we lay the basis for the calculation of x in § 3. The defining field theory is a zero-component $g_0|\boldsymbol{\phi}^2|^2$ theory. Let $\boldsymbol{\phi}(x) \in \mathbb{R}^n$ be a field variable defined at the lattice site $x \in \mathbb{Z}^d$. Define the Euclidean lattice action of the theory

$$S = -\tfrac{1}{2} \sum_{x,x' \in \mathbb{Z}^d} \left(\sum_{a=1}^{n} \phi_a(x) \gamma_{xx'} \phi_a(x') \right) + \tfrac{1}{2} r_0 \sum_{x \in \mathbb{Z}^d} \left(\sum_{a=1}^{n} \phi_a^2(x) \right) + \tfrac{1}{8} g_0 \sum_{x \in \mathbb{Z}^d} \left(\sum_{a=1}^{n} \phi_a^2(x) \right)^2 \tag{2.1}$$

where $\gamma_{xx'}$ is the nearest-neighbour coupling matrix

$$\gamma_{xx'} = \begin{cases} 1 & \text{if } |x - x'| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and (r_0, g_0) are the bare parameters of the theory. The $(n \to 0)$ limit of this theory results in a random walk theory [2] with correlation functions [1]

$$\lim_{n \to 0} \langle \phi_1(x) \phi_1(y) \rangle = \sum_{\omega : x \to y} \prod_{j \in \mathbb{Z}^d} \int \mathrm{d}\nu_{k_j(\omega)}(t_j) \exp(-r_0 t_j - \tfrac{1}{2} g_0 t_j^2) \tag{2.2}$$

where ϕ_1 is the first component of $\boldsymbol{\phi}$, ω is a random walk from x to y and

$$\mathrm{d}\nu_k = \begin{cases} \delta(t)\, \mathrm{d}t & \text{if } k = 0 \\ [t^{k-1}/(k-1)!]\theta(t)\, \mathrm{d}t & \text{if } k \geqslant 1 \end{cases} \tag{2.3}$$

and $k_j(\omega)$ is the number of times ω visits site $j \in \mathbb{Z}^d$. Expressions (2.2) and (2.3) define the so-called Edwards random walk (or self-suppressing random walk) [1-3] in a field

theoretical manner. Self-intersections of this random walk are suppressed by a potential $(-\frac{1}{2}g_0 t^2)$. The larger the value of g_0 the greater this suppression and the random walk becomes 'more' self-avoiding in character. This walk is believed to belong to the same equivalence class as the self-avoiding random walk [3], and the same values for the critical exponents are expected. It is our objective to show this explicitly.

To calculate the Hausdorf dimension of the Edwards walks is now quite straightforward; it is a simple application of the Wilson–Kogut renormalisation group theory in $(4-\varepsilon)$ dimensions [8] to (2.1):

$$1/\nu \approx 2 - \tfrac{1}{4}\varepsilon. \tag{2.4}$$

In three dimensions $\nu = \frac{4}{7}$. This is to be compared to $\nu = \frac{3}{5}$ for the self-avoiding random walk and $\nu = \frac{1}{2}$ for Brownian walks [3]. The difference in the value between the Edwards model and the self-avoiding random walk model is due to the method employed here (for a more detailed analysis see [1]).

Consider now an Edwards walk on a cubic lattice in d dimensions. This forms a network of links. If a resistor of unit resistance is assigned to each link a complicated network of resistors is formed because the random walk intersects itself on the lattice for finite values of g_0 in (2.1). The relationship between the conductivity of a network and the zero-component $(s \to 0)$ limit of an s-component Potts model has been proposed and proven by Fortuin and Kasteleyn [10]. On a lattice Λ the Hamiltonian may be written as [5]

$$-\beta H_{\mathrm{P}} = J(s-1) \sum_{\boldsymbol{q} \in \Lambda} (\boldsymbol{V}(q_1) \cdot \boldsymbol{V}(q_2) - 1) \tag{2.5}$$

where β is the inverse temperature, J is the nearest-neighbour coupling and $\boldsymbol{q}$ are the links of Λ with endpoints q_1 and q_2. $\boldsymbol{V}(x)$ is an s-state Potts vector constrained to point from the centre of an $(s-1)$-dimensional simplex to one of the s vertices, i.e.

$$\boldsymbol{V}(x) \cdot \boldsymbol{V}(x') = \begin{cases} 1 & \text{if } \boldsymbol{V}(x) \| \boldsymbol{V}(x') \\ -1/(s-1) & \text{otherwise.} \end{cases} \tag{2.6}$$

Equation (2.6) defines the partition function of this system, and it may be reduced to a density matrix (see also [6]):

$$\rho = \prod_{\boldsymbol{q} \in \Lambda} (1 + J(s-1)\boldsymbol{V}(q_1) \cdot \boldsymbol{V}(q_2)). \tag{2.7}$$

It is then relatively easy to show that the resistance $R(x, y)$ between two nodes (x, y) in Λ is related to the two-point correlation function by

$$\lim_{s \to 0} \langle \boldsymbol{V}(y) \cdot \boldsymbol{V}(x) \rangle \approx 1 - (1/J)R(x, y) + \mathrm{O}(1/J^2) - \ldots. \tag{2.8}$$

Equation (2.7) assigns a factor $(1 + J(s-1)\boldsymbol{V}(q_1) \cdot \boldsymbol{V}(q_2))$ to each link of the lattice Λ. The resistance of a random walk is calculated by assigning such a factor to each of the links of the random walk. In (2.1) the links of the random walk are formed by the nearest-neighbour coupling matrix γ indicating that in order to write down the resistance Hamiltonian, a factor such as that in (2.7) must, according to the prescription of Fortuin and Kasteleyn, be supplied for each non-zero element of γ. This factor will be replicated m times to facilitate an average over the walk, and the limit $m \to 0$ will imply $s \to 0$. For Edwards walks it is apparent that the correct theory to calculate

the resistance correlation (2.8) is

$$H = \tfrac{1}{2} \sum_{x \in \mathbb{Z}^d} \left(\sum_{a=1}^{n} \phi_a^2(x) \right) + \tfrac{1}{8} g_0 \sum_{x \in \mathbb{Z}^d} \left(\sum_{a=1}^{n} \phi_a^2(x) \right)^2 - \tfrac{1}{2} K \sum_{x,x' \in \mathbb{Z}^d} \left(\sum_{a=1}^{n} \phi_a(x) \gamma_{xx'} \phi_a(x') \prod_{\alpha=1}^{m} (1 + J(s-1) \boldsymbol{V}^\alpha(x) \cdot \boldsymbol{V}^\alpha(x')) \right). \tag{2.9}$$

The resistance correlations of this theory are then $\langle \phi_1(x) \boldsymbol{V}^1(x) \cdot \phi_1(y) \boldsymbol{V}^1(y) \rangle$ where ϕ_1 and $\boldsymbol{V}^1$ are the first components of ϕ_a and $\boldsymbol{V}^\alpha$. Indeed, by exploting the methods of [2], it is possible to show that

$$\lim_{m \to 0} \lim_{n \to 0} \langle \phi_1(x) \boldsymbol{V}^1(x) \cdot \phi_1(y) \boldsymbol{V}^1(y) \rangle$$

$$= \lim_{m \to 0} \frac{1}{Z_0} \sum_{\omega : x \to y} K^{|\omega|} \int (\mathrm{d}\boldsymbol{V}^\alpha) \left[\prod_{q \in \omega} \left(\prod_{\alpha=1}^{m} (1 + J(s-1) \boldsymbol{V}^\alpha(q_1) \cdot \boldsymbol{V}^\alpha(q_2)) \right) \right]$$

$$\times \prod_{j \in \mathbb{Z}^d} \int \mathrm{d}\nu_{k_j(\omega)}(t_j) \exp(-t_j - \tfrac{1}{2} g_0 t_j^2) \tag{2.10}$$

where ω has $|\omega|$ links, $\boldsymbol{q} \in \omega$ is a link of ω with endpoints q_1 and q_2 and $Z_0 = \lim_{n \to 0} Z$ where Z is the partition function of the theory. Each random walk is weighted by the Edwards weight, and we calculate the average resistance by expanding the correlation function in orders of $1/J$. Equation (2.10) explains the chosen form of (2.9) explicitly. The density matrix (2.7) appears in replicated form to determine the resistance of each Edwards walk as in (2.8).

The natural way is now to find an effective Hamiltonian in momentum space for (2.9) using the methods of [5, 11, 12]. This is done by assuming that g_0 is small enough for us to retain only terms to first order in it, and to fourth order in the fields. In (2.9) expand the product and put

$$f^{(t)}(a, \boldsymbol{\alpha}, \boldsymbol{i}) = (J(s-1))^{t/2} z \sqrt{K}\, \phi_a(x) V_{i_1}^{\alpha_1}(x) \ldots V_{i_t}^{\alpha_t}(x) \tag{2.11}$$

in the partition function and introduce $h^{(t)}(a, \boldsymbol{\alpha}, \boldsymbol{i})$ conjugate to $f^{(t)}(a, \boldsymbol{\alpha}, \boldsymbol{i})$. Then the partition function is

$$Z = \int (\mathrm{d}h^{(t)})(\mathrm{d}\phi_a) \sum_{\{\boldsymbol{V}^\alpha\}} \exp\left[-\tfrac{1}{2} \sum_{a=1}^{n} \phi_a^2 - \tfrac{1}{8} g_0 \left(\sum_{a=1}^{n} \phi_a^2 \right)^2 \right]$$

$$\times \exp\left(-\tfrac{1}{2} z^2 \sum_{t=0}^{m} \sum_{a=1}^{n} \sum_{\boldsymbol{\alpha}}^{m} h^{(t)}(a, \boldsymbol{\alpha}, \boldsymbol{i}) \gamma^{-1} h^{(t)}(a, \boldsymbol{\alpha}, \boldsymbol{i}) \right)$$

$$\times \exp\left(\sum_{t=0}^{m} \sum_{a=1}^{n} \sum_{\boldsymbol{\alpha}}^{m} f^{(t)}(a, \boldsymbol{\alpha}, \boldsymbol{i}) h^{(t)}(a, \boldsymbol{\alpha}, \boldsymbol{i}) \right) \tag{2.12}$$

where the lattice indices and sums over the lattice indices have been suppressed, and

$$\sum_{\boldsymbol{\alpha}}^{m} = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_t}^{m}.$$

The integral over the field ϕ_a is done by expanding the sum over the Potts fields, and by expanding the fourth-order term in the exponential as $[1 - \frac{1}{8} g_0 (\Sigma_{a=1}^{n} \phi_a^2)^2 + \mathrm{O}(g_0^2)]$ we integrate the resulting Gaussian integrals term by term. The sums over the Potts

vectors can now be evaluated, and keeping only terms to fourth order in h and first order in g_0, the Hamiltonian in momentum space is found to be

$$H_{\text{eff}} = \tfrac{1}{2}\sum_{t=0}^{m}\sum_{a=1}^{n}\int_p (r_t + p^2)h_p^{(t)}(a,\boldsymbol{\alpha},\boldsymbol{i})h_{-p}^{(t)}(a,\boldsymbol{\alpha},\boldsymbol{i})$$

$$+\tfrac{1}{8}g_1\sum_{t_1,t_2=0}^{m}\sum_{a,b=1}^{n}\sum_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{m}(Q(t_1,\boldsymbol{i},\boldsymbol{j}))(Q(t_2,\boldsymbol{k},\boldsymbol{l}))$$

$$\times\int_{p_1p_2p_3} h_{p_1}^{(t_1)}(a,\boldsymbol{\alpha},\boldsymbol{i})h_{p_2}^{(t_1)}(a,\boldsymbol{\alpha},\boldsymbol{j})h_{p_3}^{(t_2)}(b,\boldsymbol{\beta},\boldsymbol{k})h_{-p_1-p_2-p_3}^{(t_2)}(b,\boldsymbol{\beta},\boldsymbol{l})$$

$$+\tfrac{1}{8}g_2\sum_{t_1,t_2,t_3,t_4=0}^{m}\sum_{a,b=1}^{n}\sum_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta}}^{m}F(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma},\boldsymbol{\delta};\boldsymbol{i},\boldsymbol{j},\boldsymbol{k},\boldsymbol{l})$$

$$\times\int_{p_1p_2p_3} h_{p_1}^{(t_1)}(a,\boldsymbol{\alpha},\boldsymbol{i})h_{p_2}^{(t_2)}(a,\boldsymbol{\beta},\boldsymbol{j})h_{p_3}^{(t_3)}(b,\boldsymbol{\gamma},\boldsymbol{k})h_{-p_1-p_2-p_3}^{(t_4)}(b,\boldsymbol{\delta},\boldsymbol{l}) \tag{2.13}$$

where F is a coupling in the fourth-order Potts vectors as in [5] and

$$Q(t,\boldsymbol{i},\boldsymbol{j}) = \prod_{z=1}^{t} J\delta_{i_z j_z} \qquad \int_p = \frac{1}{(2\pi)^d}\int \mathrm{d}^d p \tag{2.14}$$

and where we have defined

$$r_t = [1 - g_0(1+\tfrac{1}{2}n) + \mathrm{O}(g_0^2)]z^2KJ^t + z \tag{2.15}$$

$$g_1 = [1 - g_0(2+n) + \mathrm{O}(g_0^2)]z^4K^2 \tag{2.16}$$

$$g_2 = -[1 - g_0(3+n) + \mathrm{O}(g_0^2)]z^4K^2. \tag{2.17}$$

The coupling constants in (2.9) (g_0, K, J) have now been transformed into (r_t, g_1, g_2). The lattice on which the theory had been defined is now a cutoff on the momentum. The next step is now to perform a renormalisation group transformation on (2.13). This was done on a similar Hamiltonian by Dasgupta *et al* [5] and will not be repeated here.

3. The resistance exponent

In this section the resistance exponent is calculated from the renormalisation group recursion equations. Let $g_0 = 0$ in the above. In this case we recover the Brownian random walk model studied by Banavar *et al* [6]. Clearly

$$g_1 + g_2 = 0 \tag{3.1}$$

and the recursion relations calculated to first order in $(4-\varepsilon)$ dimensions using Wilson's method (see [7, 8] for details and [6] for a comparison) are

$$g_1' = b^{\varepsilon}[g_1 - K_d \ln(b)(4g_1^2 + 2g_1g_2)] \tag{3.2}$$

$$g_2' = b^{\varepsilon}[g_2 - K_d \ln(b)(6g_1g_2 + 4g_2^2)] \tag{3.3}$$

where $K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/z)$ and b is the length scaling parameter. These equations have four fixed points, which are now briefly considered.

Figure 1 is the phase diagram for this theory. The fixed point at $(g_1^*, g_2^*) = (0, 0)$ is the Gaussian fixed point where g^* is the value of the renormalised coupling constants. It is unstable and has λ exponents $(\lambda_1, \lambda_2) = (1, 1)$. The next fixed point is at $(0, \varepsilon/4K_d)$. This is a saddle point with λ exponents $(\frac{1}{2}, -1)$. Another saddle point is found at $(\varepsilon/2K_d, -\varepsilon/2K_d)$ with λ exponents $(-2, 2)$. The only stable fixed point is found at $(\varepsilon/4K_d, 0)$ with λ exponents $(-1, -\frac{1}{2})$.

The region marked D in figure 1 has $g_1 + g_2 < 0$ and the potential of the effective Hamiltonian is non-confining. The recursion relation for r_t in the limits taken is seen to be

$$r_t' = b^2[r_t - K_d \ln(b)(r_0 g_1 + r_t g_2)]. \tag{3.4}$$

For Brownian random walks (3.1) holds and the critical behaviour of the system is selected by the saddle point at B in the figure. This implies that for their theory $1/\nu = 2 + \varepsilon/2$ and a simple scaling argument gives

$$x = \tfrac{3}{4} \tag{3.5}$$

in three dimensions. For Edwards random walks a different picture emerges. Equations (2.16) and (2.17) indicate that for sufficiently small $g_0 \neq 0$

$$|g_1| > |g_2| \tag{3.6}$$

and the typical renormalisation group theory trajectory would start at A in figure 1 and gravitate to the stable fixed point at P. The resulting critical behaviour gives a critical value of

$$x = 1 \tag{3.7}$$

for the resistance exponent for Edwards random walks. This value is, up to this order, no different from the value expected for the self-avoiding random walk.

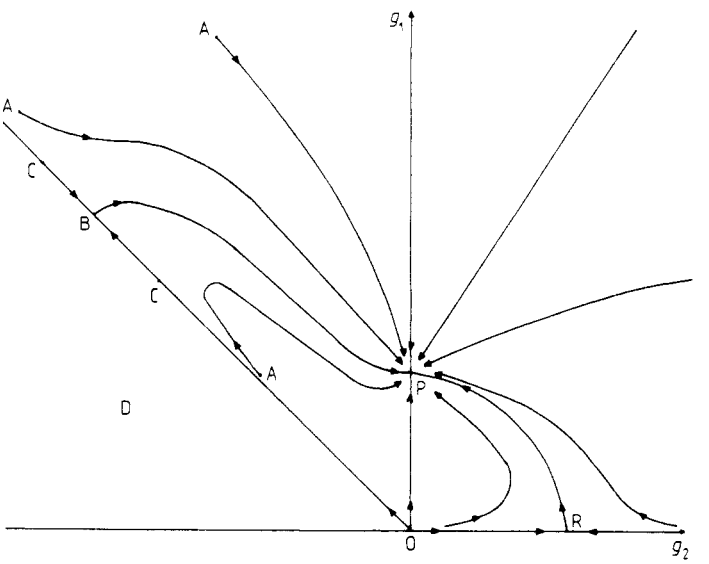

Figure 1. The renormalisation group phase diagram for resistant random walks. The region marked D has a non-confining potential. O is the Gaussian fixed point (0, 0), B the fixed point associated with resistant Brownian random walks [6] and P the fixed point for resistant Edwards random walks. A typical renormalisation group trajectory for Brownian random walks would start at C and converge to B, while for Edwards walks it starts at A and converges to P.

4. Monte Carlo algorithm for Edwards walks

The results of the previous sections suggest that the Hausdorf dimension of the Edwards random walk is $\frac{7}{4}$ and that the resistance exponent is 1 in three dimensions. In this section we shall develop a Monte Carlo (MC) algorithm [13] to generate Edwards walks and test it by generating Brownian random walks.

The goal is to generate random walks with fixed length and loose endpoints. The simulation of these kinds of objects by polymer scientists is well known [14] and Brownian walks (with fixed endpoints) were first generated in [15]. The same basic approach will be used here as in [14, 15]: a local elementary MC transition process to generate new configurations of random walks will be defined. A link of a random walk is chosen at random, which together with its nearest neighbours are dissolved and then reconstituted in an unbiased manner. This process is illustrated for a specific example in figure 2. If the selected link is on the end of the walk, then it is flipped into any of the $(2d)$ possible available positions, as in figure 3. It is obvious that this process connects configuration space.

For Brownian random walks the transition probability P is

$$P(\omega \to \omega') = 1 \tag{4.1}$$

because all configurations have the same weight in the partition function. For Edwards walks, using (2.2) and following [14], define an associated action $S(\omega)$ with each configuration ω:

$$\exp(-S(\omega)) = \prod_{j \in \omega} \int \mathrm{d}\nu_{k_j(\omega)}(t_j) \exp(-r_0 t_j - \tfrac{1}{2} g_0 t_j^2). \tag{4.2}$$

The transition probability is then

$$P(\omega \to \omega') = \begin{cases} \exp[-(S(\omega') - S(\omega))] & \text{if } S(\omega') > S(\omega) \\ 1 & \text{otherwise.} \end{cases} \tag{4.3}$$

It is then clear that the condition of detailed balance

$$\frac{P(\omega \to \omega')}{P(\omega' \to \omega)} = \exp[-(S(\omega') - S(\omega))] \tag{4.4}$$

is satisfied and we are assured of generating the correct distribution. This algorithm was coded into the ICL DAP, a 64×64 parallel processor, and one walk was stored per element giving an ensemble of 4096 random walks at any time.

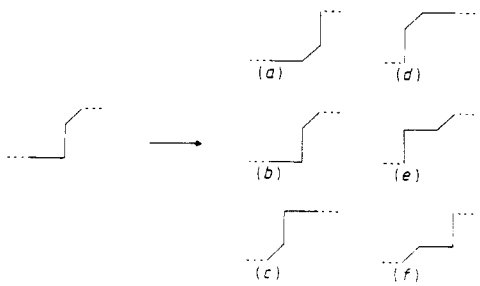

Figure 2. The elementary Monte Carlo process for the generation of Edwards random walks. The three links on the left-hand side are dissolved and reconstituted into any of (a)-(f) with equal bias.

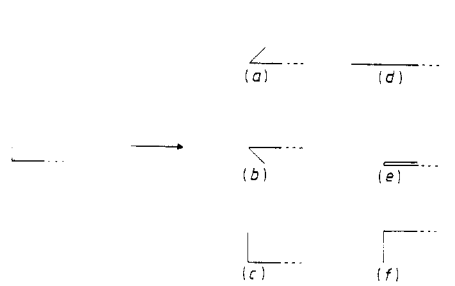

Figure 3. The elementary Monte Carlo process for a link at the end of a walk. The link is flipped into any of the positions (a)-(f).

Table 1. The correlation length ξ and the resistance R of Brownian walks. The random walks have length N and the initial ξ was measured before the ensemble was updated 1000 times with the MC algorithm to give the final measurements. A least squares fitting to the initial and final values of ξ gives $\nu = 0.496$ and $\nu = 0.497$ respectively. The resistance exponent was calculated by a least squares fitting ($x = 0.7491 \pm 0.0093$).

Length	ξ_{initial}	ξ_{final}	R
10	2.8927	2.9018	6.085 ± 0.041
20	4.1203	4.1143	10.390 ± 0.099
30	5.0475	5.0713	14.21 ± 0.13
40	5.8591	5.8305	17.66 ± 0.16
49	6.4279	6.4469	20.81 ± 0.14
60	7.1113	7.1378	23.97 ± 0.21
70	7.7132	7.7274	26.92 ± 0.25
80	8.2734	8.2533	—

To test the algorithm, an ensemble of Brownian walks (already in equilibrium) was operated on with 1000 iterations on each. A new ensemble of Brownian walks is expected. Of these random walks, the correlation length (square root mean square end-to-end distance) ξ was measured before and after, and the electric resistance was measured after the application of the algorithm. From the result in table 1, it is apparent that the algorithm leaves an ensemble of Brownian walks unperturbed. To calculate critical exponents from these data corrections to scaling were taken into account by choosing the scaling laws

$$\xi \approx (N+k)^{\nu} \tag{4.5}$$

$$R \approx (N+k)^{x}. \tag{4.6}$$

These forms should be compared to those chosen by Beretti and Sokal [14]. A numerical method was used to perform a least squares fitting of the data to (4.5) and (4.6) (Newton's method). The results are

$$\begin{aligned} \xi_{\text{initial}} &= 0.9397(N-0.3466)^{0.4961} \\ \xi_{\text{final}} &= 0{\cdot}9346(N-0{\cdot}2447)^{0{\cdot}4973} \\ R &= 1{\cdot}126(N-0{\cdot}4995)^{0{\cdot}7491\pm 0{\cdot}0093} \end{aligned} \tag{4.7}$$

These results are to be compared with the expected values of $\nu = \frac{1}{2}$ and $x = \frac{3}{4}$ [6] for a Brownian random walk in three dimensions. The error bar in x is two standard deviations. The above quantities were obtained by a least squares fitting to the data in table 1 and the error in x was calculated using standard procedures [16]. The resistance of each random walk was calculated by the exact solution of Kirchhoff's laws at every node of the network by using a standard Gauss-Jordan elimination scheme. The measured quantities are averages over 4096 walks each.

5. Numerical results

In this section the critical exponents ν and x are calculated for the Edwards random walk. The expected numerical values are now already known from the considerations in §§ 2 and 3: $\nu = 0{\cdot}571$ and $x = 1$.

The MC algorithm developed in the previous section was used to generate ensembles of Edwards walks. As an initial ensemble, Brownian walks of length N were chosen and operated upon to become Edwards walks. The study in §§ 2 and 3 was based on perturbation theory, so a value of g_0 in the perturbative region $g_0 = \frac{1}{2}$ was chosen at first. The value of r_0 (4.2) was set equal to 1, and the scaling of the walks was studied by making N large. As an example in the non-perturbative region, the value of g_0 was chosen to be 2. The results for values of N from $N = 10$ to $N = 50$ are displayed in tables 2 and 3. A plot of ln R against ln N suggests scaling (see figure 4) in three dimensions. The critical exponents were calculated by least squares fits to the data using the scaling forms (4.5) and (4.6) and the errors were once again calculated in the standard fashion [16]. For $g_0 = \frac{1}{2}$ the results are

$$\xi = 0{\cdot}8854(N - 1{\cdot}202)^{0{\cdot}551 \pm 0{\cdot}012} \qquad R = 0{\cdot}5842(N + 3.226)^{0{\cdot}978 \pm 0{\cdot}010} \tag{5.1}$$

where the fitting for ξ is over walks of length 15-50 and for R over walks to length 10-50. For $g_0 = 2$

$$\xi = 0{\cdot}9381(N - 0{\cdot}5722)^{0{\cdot}561 \pm 0{\cdot}010} \qquad R = 0{\cdot}7903(N + 1{\cdot}359)^{0{\cdot}983 \pm 0{\cdot}010} \tag{5.2}$$

over the same lengths as for $g_0 = \frac{1}{2}$. The errors are two standard deviations.

The results for ν and x are in excellent agreement with the results of the perturbation theory calculation. The value of k in (4.5) and (4.6) proved to be small compared with N for all the least squares fittings done, so we are confident that the scaling forms are justified.

Table 2. The correlation length ξ and resistance R for Edwards random walks with $g_0 = \frac{1}{2}$ and $r_0 = 1$. A least squares fitting gives $\nu = 0.551 \pm 0.012$ and $x = 0.978 \pm 0.010$.

Length	Iterations	ξ	R
10	10 000	3.353 ± 0.020	7.421 ± 0.037
15	15 000	3.812 ± 0.024	9.714 ± 0.053
20	20 000	4.307 ± 0.027	12.555 ± 0.069
25	25 000	5.214 ± 0.032	15.524 ± 0.079
30	30 000	5.723 ± 0.037	18.023 ± 0.093
34	35 000	—	20.533 ± 0.103
35	35 000	6.159 ± 0.039	—
40	80 000	6.653 ± 0.042	23.639 ± 0.117
45	85 000	7.065 ± 0.045	25.941 ± 0.129
50	120 000	7.295 ± 0.046	27.816 ± 0.142

Table 3. The correlation length ξ and the resistance R of Edwards walks with $g_0 = 2$ and $r_0 = 1$. These data give $\nu = 0.561 \pm 0.010$ and $x = 0.983 \pm 0.010$.

Length	Iterations	ξ	R
10	10 000	3.619 ± 0.020	8.669 ± 0.029
15	15 000	4.243 ± 0.024	12.214 ± 0.044
20	20 000	4.811 ± 0.028	16.030 ± 0.055
25	25 000	5.730 ± 0.034	19.624 ± 0.069
30	30 000	6.349 ± 0.038	23.663 ± 0.080
35	35 000	6.853 ± 0.043	27.391 ± 0.094
40	80 000	7.296 ± 0.044	30.689 ± 0.105
45	85 000	7.955 ± 0.050	34.420 ± 0.116
50	90 000	8.316 ± 0.051	37.664 ± 0.127

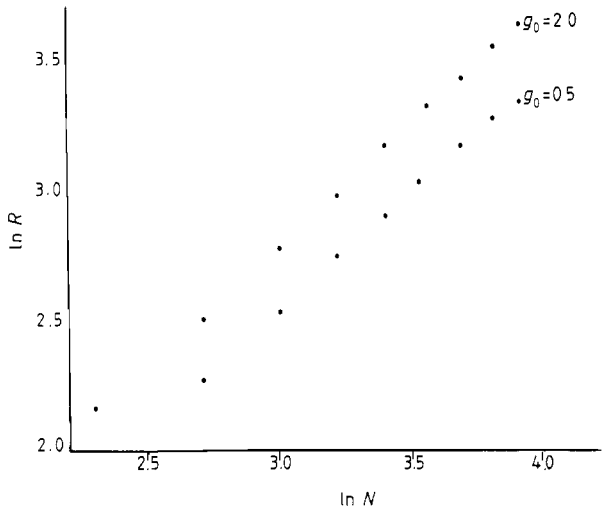

Figure 4. Scaling for resistant Edwards walks. The resistance exponent x is calculated from the slopes of the curves by a least squares fitting.

The measured quantities are averages over 4096 walks each. The program could perform up to 30 000 iterations per hour and ensembles of walks containing 50 links were relaxed for up to 4 h before any readings were taken to ensure the correct distribution of Edwards walks. In total, the calculation took 34 h to equilibrate the ensembles and 14 h to calculate the resistance of the ensembles.

6. Conclusions

The results that we obtained for the Brownian random walk in this paper should be compared with the results by Banavar *et al* [6]. Using a renormalisation group scheme, the value of x was found to be $\frac{3}{4}$ and numerically 0.73 ± 0.005. This compares well with the value in this paper. Ball and Cates [9] determined the resistance of self-avoiding random walk clusters with all nearest-neighbour sites connected by unit resistors. They found, even in that model, the value of x to be 1. That result implies that the model described here shares the same characteristics as theirs: the random network contains so many bottlenecks that typically the end-to-end circuit will be disrupted by cutting a single link of the chain.

For the Edwards model, we believe that the calculation by perturbative methods of the resistance exponent was successful, and this is supported by the results obtained numerically. Although the analysis here is perturbative, the structure of the renormalisation flow diagram (figure 1) suggests that the results obtained are valid to all orders in the ε expansion.

The calculation of larger moments of the resistance correlation

$$R^n \propto N^{x_n} \tag{6.1}$$

where R^n is the nth moment of the average resistance over all configurations, both by numerical and analytic means (see [6]), is still outstanding for the Brownian as well as the Edwards model. Another outstanding result concerns the scaling of the shortest

arc length connecting the endpoints of the random walk cluster. In the light of the results obtained so far it would seem to scale with the same value of the critical exponent as the resistance (1).

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